

GENERALISED HYPERBOLICITY IN CONICAL SPACE-TIMES

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ABSTRACT. Solutions of the wave equation in a space-time containing a thin cosmic string are examined in the context of non-linear generalised functions. Existence and uniqueness of solutions to the wave equation in the Colombeau algebra \mathcal{G} is established for a conical space-time and this solution is shown to be associated to a distributional solution. A concept of generalised hyperbolicity, based on test fields, can be defined for such singular space-times and it is shown that a conical space-time is \mathcal{G} -hyperbolic.

1. INTRODUCTION

Weak singularities have for some time been used to model physically plausible scenarios such as thin cosmic strings, impulsive gravitational waves and shell crossing singularities. Such singularities typically admit a locally bounded metric and well behaved curvature scalars but none the less are still classified as singularities due to a low differentiable metric resulting from topological defects.

One such example is the conical singularity, used to model thin cosmic strings, resulting from writing down Minkowski space-time in cylindrical coordinates

$$ds^2 = -dt^2 + dr^2 + r^2 d\phi^2 + dz^2$$

and identifying $\phi = 0$ with $\phi = 2\pi A$ for $A < 1$.

The singular behaviour of this space time becomes apparent when one parallelly transports a frame around a closed curve around the axis Λ and finds that it undergoes a holonomy of the same magnitude as the deficit angle $2\pi(1 - A)$. For non-singular space-times with a C^{2-} regular metric it may be shown that such holonomies are trivial (See Wilson and Clarke (1996) for a proof); it is therefore the case that there is a problem with differentiability. In suitable Cartesian coordinates

$$x^0 = t, \quad x^1 = r \cos(\phi/A), \quad x^2 = r \sin(\phi/A), \quad x^3 = z$$

the metric may be written as

$$\begin{aligned} ds^2 = & -dt^2 + dz^2 + \frac{1}{2}(1 + A^2)((dx^1)^2 + (dx^2)^2) \\ & + \frac{1}{2}(1 - A^2)\left(\frac{(x^1)^2 - (x^2)^2}{(x^1)^2 + (x^2)^2}\right)((dx^1)^2 - (dx^2)^2) \\ & + \frac{1}{2}(1 - A^2)2\left(\frac{2x^1x^2}{(x^1)^2 + (x^2)^2}\right)dx^1 dx^2 \end{aligned} \quad (1)$$

which, although it is locally bounded, admits directional dependent limits as one approaches the axis.

The low differentiability of the metric at the axis does not prevent the curvature being calculated as a distribution. One can not directly use conventional distribution theory techniques because this would involve the evaluation of ill defined products of distributions which are not well defined (Schwartz, 1954). Geroch and Traschen (1987) introduced a class of regular metrics for which the distributional curvature is guaranteed to be well defined and showed that such a space-time must have distributional curvature with support on a submanifold of at most co-dimension one, which is clearly not the case for the cone. Instead regularisation techniques may be used such as those employed by Balasin and Nachbagauer (1993) and especially those using the framework of Colombeau's non-linear generalised function theory (Colombeau, 1984) such as Clarke et al., (1996). Using such techniques it may be shown that the non-zero components of the energy-momentum tensor density are

$$T^0_0(-g)^{1/2} = T^3_3(-g)^{1/2} = -2\pi(1 - A)\delta^{(2)}(x^1, x^2)$$

This form of the energy momentum tensor is what justifies the use of this kind of space-time to model a thin cosmic string. This suggests that the interpretation of the curvature as a distribution could be one such criterion for regarding these quasi-regular singularities as physically plausible.

Another question relating to the physical importance of a singularity arises out of the question of to what extent the singularity disrupts the evolution of Einstein's equations and therefore global hyperbolicity. For space-times with a C^{2-} metric (which guarantees the existence of unique geodesics) this question is answered by requiring the space-time to be globally hyperbolic (Penrose, 1979). There are however a number of space-times with a lower differentiability for which, although they may violate cosmic censorship, there still may be well-posed initial-value problems for test fields. This led to the proposal of the concept of generalised hyperbolicity (Clarke, 1998) in which one examined the extent to which singularities were obstructions to the evolution of the wave equation. In order to apply this concept to space-times with shell crossing singularities Clarke replaced the initial value problem for the wave equation

$$\square u = f, \quad u|_S = v, \quad (n^a \partial_a u)|_S = w$$

by a distributional version obtained by multiplying by a test field ψ and integrating by parts once (rather than twice as is usual) to give

$$\int_{M^+} \partial_a u \partial_b \psi g^{ab} \mu = - \int_{M^+} \psi f \mu - \int_S \psi w \mu_S$$

He then said that a space-time was \square -globally hyperbolic if the above equation had a unique solution for all $\psi \in \mathcal{D}(M)$. Moreover it was shown that this form of generalised hyperbolicity was satisfied for a class of curve-integrable space-times and in particular it was demonstrated for the shell-crossing dust space-times.

In this paper we shall consider the question of generalised hyperbolicity for the conical space-time (1). Unfortunately this space-time does not admit a locally square integrable connection, so we are unable to apply the result of Clarke (1998) directly. Instead we shall follow a different approach: We shall use Colombeau's generalised functions to overcome the ambiguities which arise when attempting to multiply distributions which would arise when considering solutions to the wave equation in a space-time of such low differentiability. This involves writing down the Cauchy problem for wave equation in the Colombeau algebra $\mathcal{G}(M)$, and proving the existence of a unique generalised function solution. This will be done for a class of singular metrics which includes the 4-cone (1) (Section 4). Having obtained a unique solution we shall examine to what extent it is possible to interpret it as a distribution. We say that a space-time is \mathcal{G} -globally hyperbolic if there exists a unique solution to the wave equation in the Colombeau algebra \mathcal{G} and that this solution is associated to a distribution. We therefore demonstrate in this paper that a conical space-time is \mathcal{G} -hyperbolic.

2. COLOMBEAU'S GENERALISED FUNCTIONS

We first briefly recall the essential details of Colombeau's theory (see e.g. Colombeau, 1984 or Oberguggenberger, 1992 for further details). We denote the space of smooth functions with compact support $\mathcal{D}(\mathbb{R}^n)$, so that the space of distributions is $\mathcal{D}'(\mathbb{R}^n)$ and we define the following spaces;

Definition 1. We define the space of smoothing kernels $\mathcal{A}_q(\mathbb{R}^n)$ as the space of all functions $\varphi^{(n)} \in \mathcal{D}(\mathbb{R}^n)$ such that

$$\begin{aligned} \int_{\mathbb{R}^n} \varphi^{(n)}(\xi) d^n \xi &= 1 \\ \int_{\mathbb{R}^n} \varphi^{(n)}(\xi) \xi^k d^n \xi &= 0 \quad \forall k \in \mathbb{N}^n \quad \text{such that} \quad 1 \leq |k| \leq q \end{aligned}$$

where we are using a multi-index notation so that $\xi^k = (\xi^1)^{k_1} \dots (\xi^n)^{k_n}$.

Given a function $f \in L^1_{\text{Loc}}(\mathbb{R}^n)$, we may smooth it by constructing the following convolution;

$$\tilde{f}(\varphi_\varepsilon, x) = \int_{\mathbb{R}^n} f(\xi) \varphi_\varepsilon^{(n)}(\xi - x) d^n \xi$$

where

$$\varphi_\varepsilon^{(n)}(\xi) = \varepsilon^{-n} \varphi^{(n)}(\xi/\varepsilon)$$

and for more general distributions $T \in \mathcal{D}'(\mathbb{R}^n)$ we have

$$\tilde{T}(\varphi_\varepsilon, x) = \langle T, \tau_x \varphi_\varepsilon^{(n)} \rangle$$

where

$$\tau_x \varphi^{(n)}(\xi) = \varphi^{(n)}(\xi - x)$$

We would like to construct a differential algebra of such smoothings; We define a base space $\mathcal{E}(\mathbb{R}^n)$ as the space of all functions $\tilde{f} : \mathcal{A}_0 \times \mathbb{R}^n \rightarrow \mathbb{C}$ which are smooth as a function of x . Certainly smoothings of distributions are elements of this space; however one desirable property would be for the smoothing of $f \in C^\infty$ to coincide with its natural embedding in that

$$\tilde{f}(\varphi_\varepsilon, x) = \int f(x) \varphi_\varepsilon^{(n)}(\xi - x) d^n \xi$$

and

$$\hat{f}(\varphi_\varepsilon, x) = f(x)$$

are equivalent. This is achieved by constructing a space of null functions $\mathcal{N}(\mathbb{R}^n)$ such that $\tilde{f} - \hat{f} \in \mathcal{N}(\mathbb{R}^n)$ and by working with a quotient space $\mathcal{E}(\mathbb{R}^n)/\mathcal{N}(\mathbb{R}^n)$.

Definition 2. The space of null functions $\mathcal{N}(\mathbb{R}^n)$ consists of the functions $\tilde{f} \in \mathcal{E}(\mathbb{R}^n)$ such that given $K \subset \subset \mathbb{R}^n$ and any collection of indices p_1, \dots, p_k with $k \in \mathbb{N}$ there is some $N \in \mathbb{N}$ and an increasing unbounded sequence (γ_q) such that for each $\varphi \in \mathcal{A}_q$ for $q \geq N$, $\exists c, \eta > 0$ such that

$$\sup_{x \in K} |\partial_{p_1} \dots \partial_{p_k} \tilde{f}(\varphi_\varepsilon, x)| \leq c \varepsilon^{\gamma_q - N} \quad (0 < \varepsilon < \eta)$$

The quotient space $\mathcal{E}(\mathbb{R}^n)/\mathcal{N}(\mathbb{R}^n)$ is not well defined because $\mathcal{N}(\mathbb{R}^n)$ is not an ideal of $\mathcal{E}(\mathbb{R}^n)$. We may multiply a null function with a function of non-polynomial growth in ε^{-1} to give a function which is not null. This problem is rectified by restricting $\mathcal{E}(\mathbb{R}^n)$ to a space of moderate functions $\mathcal{E}_M(\mathbb{R}^n)$ which itself is an algebra and of which $\mathcal{N}(\mathbb{R}^n)$ is an ideal. It may be verified that the smoothings of distributions are indeed moderate.

Definition 3. The space of moderate functions $\mathcal{E}_M(\mathbb{R}^n)$ consist of the functions $\tilde{f} \in \mathcal{E}(\mathbb{R}^n)$ such that given $K \subset \subset \mathbb{R}^n$ and any collection of indices p_1, \dots, p_k with $k \in \mathbb{N}$ there is some $N \in \mathbb{N}$ such that for each $\varphi \in \mathcal{A}_N$, $\exists c, \eta > 0$ such that

$$\sup_{x \in K} |\partial_{p_1} \dots \partial_{p_k} \tilde{f}(\varphi_\varepsilon, x)| \leq c \varepsilon^{-N} \quad (0 < \varepsilon < \eta)$$

We finally define our space of generalised functions as the quotient space

$$\mathcal{G}(\mathbb{R}^n) = \frac{\mathcal{E}_M(\mathbb{R}^n)}{\mathcal{N}(\mathbb{R}^n)}$$

The space $\mathcal{G}(\mathbb{R}^n)$ is a differential algebra, of which $C^\infty(\mathbb{R}^n)$ is a subalgebra and $\mathcal{D}'(\mathbb{R}^n)$ is a linear subspace. In practice one usually works with the representative moderate functions to perform calculations in this space. Many elements of $\mathcal{G}(\mathbb{R}^n)$ do not result as the smoothing of distributions, but even so are equivalent to a distribution in the following sense;

Definition 4. We say that $[\tilde{f}] \in \mathcal{G}(\mathbb{R}^n)$ has an associated distribution $T \in \mathcal{D}'(\mathbb{R}^n)$ (written as $[\tilde{f}] \approx T$) if for any representative \tilde{f} and $\psi \in \mathcal{D}(\mathbb{R}^n)$ there exists some $N \in \mathbb{N}$ such that

$$\lim_{\varepsilon \rightarrow 0} \int \tilde{f}(\varphi_\varepsilon, x) \psi(x) d^n x = \langle T, \psi \rangle$$

It may be shown, for example, that if \tilde{H} and $\tilde{\delta}$ are the smoothed Heaviside and delta distributions respectively then $[\tilde{H}]^2 \approx H$ and $[\tilde{H}][\tilde{\delta}] \approx \frac{1}{2}\delta$. On the other hand, the generalised function $[\tilde{\delta}]^2$ has no associated distribution.

When one applies the theory to initial value problems one usually makes the additional requirement that the n -dimensional smoothing kernels are Cartesian products of one dimensional kernels (Oberguggenberger 1992) so that

$$\varphi^{(n)}(x^1, \dots, x^n) = \varphi^{(1)}(x^1) \cdots \varphi^{(1)}(x^n).$$

The reason for doing this is that it gives a natural way of defining Cartesian products and restrictions. However in order to simplify the calculations in Section 5 it is useful to work with rotationally symmetric smoothing kernels. Unfortunately it is not possible to write such a kernel as a finite sum of Cartesian products and for this reason we describe in Appendix A an alternative way of restricting 4-dimensional smoothing kernels to an initial surface S so that any spatial symmetries are preserved. Note however that the results of the next two sections are not sensitive to the precise restriction process adopted.

3. THE WAVE EQUATION IN THE COLOMBEAU ALGEBRA $\mathcal{G}(M)$

We first introduce the convention that lower case Latin indices a, b, \dots run over $0 \dots 3$, Greek indices α, β, \dots run over $1 \dots 3$ and upper case indices A, B, \dots run over 1 and 2. Suppose we have a space-time, equipped with a locally bounded singular metric, (M, g) and we wish to solve the Cauchy problem for the wave equation

$$\begin{aligned}\square u(t, x^\alpha) &= 0 \\ u(0, x^\alpha) &= v(x^\alpha) \\ \partial_t u(0, x^\alpha) &= w(x^\alpha)\end{aligned}$$

where initial data (v, w) lying in the Sobolev spaces $\mathcal{H}^1(S) \times \mathcal{H}^0(S)$ is prescribed on the initial surface $S, t = 0$. If a solution exists one would expect it to be defined as a distribution. This however will cause difficulty in interpreting

$$\square u = (-g)^{-1/2} \partial_a ((-g)^{1/2} g^{ab} \partial_b u)$$

as a distribution in the framework of Schwartzian distribution theory because it contains products of u with a singular metric and its weak derivatives which are not well defined in this theory. The way to overcome these shortcomings is to employ the non-linear generalised function theory of Colombeau (1984), which does give a distributional interpretation to many distributional products that would otherwise be undefined. We first embed the metric g_{ab} into the generalised function space $\mathcal{G}(M)$ by constructing its representative $(g_{ab}^\varepsilon) \in \mathcal{E}_M(M)$ as the convolution integral

$$g_{ab}^\varepsilon(t, x^\alpha) = \int g_{ab}(t + \varepsilon \zeta, x^\alpha + \varepsilon \xi^\alpha) \varphi^{(4)}(\zeta, \xi^\alpha) d\zeta d^3 \xi$$

Since the initial data (v, w) does not have to be smooth, we must also embed it into the space $\mathcal{G}(S) \times \mathcal{G}(S)$ as (V, W) by defining the convolution integrals v_ε and w_ε in $\mathcal{E}_M(S)$;

$$\begin{aligned}v_\varepsilon(x^\alpha) &= \int v(x^\alpha + \varepsilon \xi^\alpha) \varphi^{(3)}(\xi^\alpha) d^3 \xi \\ w_\varepsilon(x^\alpha) &= \int w(x^\alpha + \varepsilon \xi^\alpha) \varphi^{(3)}(\xi^\alpha) d^3 \xi\end{aligned}$$

The Generalised function wave operator $\square : \mathcal{G}(M) \rightarrow \mathcal{G}(M)$ then may be defined by denoting $\square U$ for $U \in \mathcal{G}(M)$ as the class represented by $(\square^\varepsilon u_\varepsilon) \in \mathcal{E}_M(M)$ where $(u_\varepsilon) \in \mathcal{E}_M(M)$ is a representative for U and

$$\square^\varepsilon u_\varepsilon = (-g_\varepsilon)^{-1/2} \partial_a ((-g_\varepsilon)^{1/2} g_\varepsilon^{ab} \partial_b u_\varepsilon)$$

We would like to then be able to solve the Cauchy problem in the space $\mathcal{G}(M)$

$$\begin{aligned}\square U(t, x^\alpha) &= 0 \\ U(0, x^\alpha) &= V(x^\alpha) \\ \partial_t U(0, x^\alpha) &= W(x^\alpha)\end{aligned} \tag{2}$$

and obtain a solution $U \in \mathcal{G}(M)$ which is associated to a distribution. In practice one works with the equivalent problem in $\mathcal{E}_M(M)$;

$$\begin{aligned}\square^\varepsilon u_\varepsilon(t, x^\alpha) &= f_\varepsilon(t, x^\alpha) \\ u_\varepsilon(0, x^\alpha) &= v_\varepsilon(x^\alpha) \\ \partial_t u_\varepsilon(0, x^\alpha) &= w_\varepsilon(x^\alpha)\end{aligned} \tag{3}$$

where $(f_\varepsilon) \in \mathcal{N}(M)$ and obtain the associated distribution (if there is one) u by evaluating the limit for each $\psi \in \mathcal{D}(M)$

$$\langle u, \psi \rangle = \lim_{\varepsilon \rightarrow 0} \int u_\varepsilon(\varphi, (t, x^\alpha)) \psi(t, x^\alpha) \mu^\varepsilon(t, x^\alpha) \quad (4)$$

where $\varphi \in \mathcal{A}_q$ for $q \in \mathbb{N}$ large enough.

The question of existence and uniqueness of solutions to (2) is not as straight-forward as for its classical function counterpart. Although one may establish the existence of unique solutions for (3) (for each $\varphi \in \mathcal{A}_0$ and $\varepsilon > 0$) by using the Cauchy-Kowalewska theorem for the case of analytic data, this is not enough to establish the existence of unique solutions to (2). One must carefully examine the role that moderate and null functions have to play. (See Oberguggenberger (1989) for examples of similar Cauchy problems that do not admit unique solutions). Firstly, we want U to be an element of $\mathcal{G}(M)$. This means that we have to verify that $(u_\varepsilon) \in \mathcal{E}_M(M)$. Secondly for U to be a unique we only require that the solution (u_ε) of (3) are unique up to an element of $\mathcal{N}(M)$. This means that for a particular choice of (f_ε) , (v_ε) and (w_ε) we are allowed to have more than one solution provided they are all well defined elements of $\mathcal{E}_M(M)$ and differ by elements of $\mathcal{N}(M)$. We must also take into account that (f_ε) , (v_ε) and (w_ε) could have been chosen with freedom up to a null function, so we require that the solution (u_ε) is augmented by at most a null function if (f_ε) , (v_ε) and (w_ε) are augmented by null functions. Since this Cauchy problem is linear it follows that the solution $U \in \mathcal{G}(M)$ is unique if for the problem (3), with $(v_\varepsilon), (w_\varepsilon) \in \mathcal{N}(S)$ implies that $(u_\varepsilon) \in \mathcal{N}(M)$.

Moreover even if there is a unique solution $U \in \mathcal{G}(M)$, it need not admit an associated distribution because the limit (4) may be dependent on the smoothing kernel $\varphi \in \mathcal{A}_q$ or may not even be defined, however large we may choose q .

4. PROOF OF THE EXISTENCE OF UNIQUE SOLUTIONS IN $\mathcal{G}(M)$

We start by observing that we may take a representation in $\mathcal{E}_M(M)$ of the embedded metric which admits the form;

$$ds_\varepsilon^2 = -dt^2 + dz^2 + g_{AB}^\varepsilon(x^C) dx^A dx^B \quad (5)$$

where $g_{AB}^\varepsilon \in \mathcal{E}_M(M)$ and locally g_{AB}^ε and its derivatives may be bounded as follows;

$$\begin{aligned} |g_{AB}^\varepsilon| &\leq M_0 \\ |g_\varepsilon^{AB}| &\leq M_0 \\ |\partial_{C_1} \dots \partial_{C_k} g_{AB}^\varepsilon| &\leq \frac{M_k}{\varepsilon^k} \end{aligned}$$

with M_k being positive constants independent of ε . It is easily seen that on embedding the conical metric (1) into $\mathcal{G}(M)$, the g_{AB}^ε and its derivatives admit this form;

$$\begin{aligned} g_{11}^\varepsilon(x^A) &= \frac{1}{2}(1 + A^2) + \frac{1}{2}(1 - A^2)P_\varepsilon(x^A) \\ g_{12}^\varepsilon(x^A) &= \frac{1}{2}(1 - A^2)Q_\varepsilon(x^A) \\ g_{22}^\varepsilon(x^A) &= \frac{1}{2}(1 + A^2) - \frac{1}{2}(1 - A^2)P_\varepsilon(x^A) \end{aligned}$$

where

$$\begin{aligned} P_\varepsilon(x^A) &= \frac{1}{\varepsilon^2} \int \frac{(\xi^1)^2 - (\xi^2)^2}{(\xi^1)^2 + (\xi^2)^2} \varphi^{(2)}\left(\frac{\xi^A - x^A}{\varepsilon}\right) d^2\xi \\ Q_\varepsilon(x^A) &= \frac{1}{\varepsilon^2} \int 2 \frac{\xi^1 \xi^2}{(\xi^1)^2 + (\xi^2)^2} \varphi^{(2)}\left(\frac{\xi^A - x^A}{\varepsilon}\right) d^2\xi \end{aligned}$$

and $\varphi^{(2)}$ is a two dimensional smoothing kernel obtained by integrating out the t and z dependence.

Our aim is to estimate solutions u_ε of (3) and its derivatives in terms of powers of ε , given the moderate and null bounds of f_ε , v_ε and w_ε using a method of energy estimates following Hawking and Ellis (1973) and Clarke (1998). We shall assume that K is a compact region of M which intersects the initial hypersurface $S = S_0$ described by $t = 0$ and that and $\{S_\tau\}_{0 \leq \tau \leq \tau_1}$ is a family of smooth space-like hypersurfaces in K which intersect S_0 on a common 2-surface. See Figure 1.

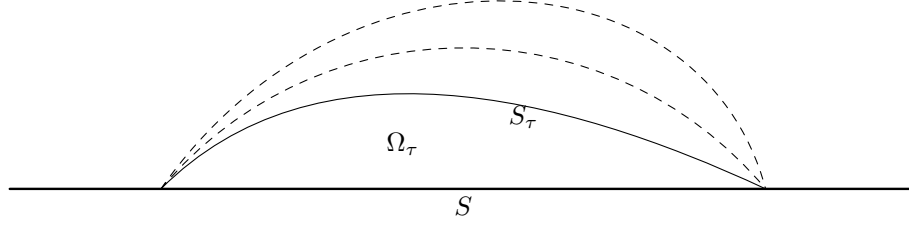


Figure 1.

We shall assume that the surfaces S_τ may be expressed by the equation $\sigma(t, x^\alpha) = \tau$; we may therefore define a normal to S_τ , $n_a = \partial_a \tau$ and a surface element $\mu_{S_\tau}^\varepsilon$ which satisfies

$$\mu_\varepsilon = (-g_\varepsilon)^{1/2} d^4 x = \mu_{S_\tau}^\varepsilon \wedge d\tau$$

We shall write the region bounded by S_0 and S_τ as

$$\Omega_\tau = \bigcup_{\zeta \in [0, \tau]} S_\zeta$$

Since we will have to estimate u_ε and its derivatives of arbitrary order, we shall construct a hierarchy of energies. It will turn out that it is much more convenient to work with energy integrals based on covariant derivatives rather than classical Sobolev norms based on partial differentiation. We define our energy integrals by the formula

$$E_\varepsilon^k(\tau) = \sum_{j=0}^k \int_{S_\tau} T_{\varepsilon,j}^{ab} t_b n_a \mu_{S_\tau}^\varepsilon \quad (6)$$

where

$$\begin{aligned} T_{\varepsilon,k}^{ab} &= (g_\varepsilon^{ac} g_\varepsilon^{bd} - \frac{1}{2} g_\varepsilon^{ab} g_\varepsilon^{cd}) e_\varepsilon^{p_1 q_1} \dots e_\varepsilon^{p_{k-1} q_{k-1}} \nabla_c^\varepsilon \nabla_{p_1}^\varepsilon \dots \nabla_{p_{k-1}}^\varepsilon u_\varepsilon \nabla_d^\varepsilon \nabla_{q_1}^\varepsilon \dots \nabla_{q_{k-1}}^\varepsilon u_\varepsilon \\ T_{\varepsilon,0}^{ab} &= -\frac{1}{2} g_\varepsilon^{ab} u_\varepsilon^2 \\ t_a &= \partial_a t \\ e_\varepsilon^{ab} &= g_\varepsilon^{ab} + 2t^a t^b \end{aligned}$$

The form of the metric (5) guarantees that $\nabla_a^\varepsilon t_b = 0$. This enables the tensor e_ε^{ab} to be constructed so that it is positive definite and is annihilated by covariant differentiation i.e. $\nabla_a^\varepsilon e_\varepsilon^{bc} = 0$.

We also define the following 3-dimensional Sobolev norms

$$\begin{aligned} \|u_\varepsilon\|_{S_\tau}^k &= \left\{ \sum_{\substack{p_1 \dots p_j \\ 0 \leq j \leq k}} \int_{S_\tau} |\partial_{p_1} \dots \partial_{p_j} u_\varepsilon|^2 \mu_{S_\tau}^\varepsilon \right\}^{1/2} \\ \|u_\varepsilon\|_{S_\tau}^{\tilde{k}} &= \left\{ \sum_{\substack{p_1 \dots p_j \\ 0 \leq j \leq k}} \int_{S_\tau} |\nabla_{p_1}^\varepsilon \dots \nabla_{p_j}^\varepsilon u_\varepsilon|^2 \mu_{S_\tau}^\varepsilon \right\}^{1/2} \end{aligned}$$

These are equivalent to the energy integrals in the following sense

Lemma 1. *There exist positive constants A_k , A'_k , B_k and B'_k such that*

$$E_\varepsilon^k(\tau) \leq A_k (\|u_\varepsilon\|_{S_\tau}^k)^2 \quad (7)$$

$$(\|u_\varepsilon\|_{S_\tau}^{\tilde{k}})^2 \leq A'_k E_\varepsilon^k(\tau) \quad (8)$$

$$(\|u_\varepsilon\|_{S_\tau}^{\tilde{k}})^2 \leq B'_k \sum_{j=1}^k \frac{1}{\varepsilon^{2(k-j)}} (\|u_\varepsilon\|_{S_\tau}^j)^2 \quad (9)$$

$$(\|u_\varepsilon\|_{S_\tau}^k)^2 \leq B_k \sum_{j=1}^k \frac{1}{\varepsilon^{2(k-j)}} (\|u_\varepsilon\|_{S_\tau}^{\tilde{j}})^2 \quad (10)$$

$\|u_\varepsilon\|_{S_\tau}^k$, $\|u_\varepsilon\|_{S_\tau}^{\tilde{k}}$ and $E_\varepsilon^k(\tau)$ are functions of τ and φ . We now define the concepts of having a moderate or null bound for such an object.

Definition 5. We say that $\rho : \mathcal{A}_0 \rightarrow \mathbb{C}$ has a moderate bound if there is some $N \in \mathbb{N}$ such that for each $\varphi \in \mathcal{A}_N$, $\exists c, \eta > 0$ such that

$$|\rho(\varphi_\varepsilon)| \leq c\varepsilon^{-N} \quad (0 < \varepsilon < \eta)$$

Definition 6. We say that $\rho : \mathcal{A}_0 \rightarrow \mathbb{C}$ has a null bound if there is some $N \in \mathbb{N}$ and an increasing unbounded sequence (γ_q) such that for each $\varphi \in \mathcal{A}_q$ for $q \geq N$, $\exists c, \eta > 0$ such that

$$|\rho(\varphi_\varepsilon)| \leq c\varepsilon^{\gamma_q - N} \quad (0 < \varepsilon < \eta)$$

An important consequence of Lemma 1 is that $E_\varepsilon^k(\tau)$ has a moderate bound if and only if $\|u_\varepsilon\|_{S_\tau}^k$ has a moderate bound and that $E_\varepsilon^k(\tau)$ has a null bound if and only if $\|u_\varepsilon\|_{S_\tau}^k$ has a null bound. It is therefore sufficient to estimate these energy integrals. The idea is to form an energy inequality to give a bound on $E_\varepsilon^k(\tau)$ in terms of the lower order energies $E_\varepsilon^1(\tau), \dots, E_\varepsilon^{k-1}(\tau)$ and $E_\varepsilon^1(0), \dots, E_\varepsilon^k(0)$, which is determined by the initial data, and to show that the properties of the $E_\varepsilon^k(0)$ having a moderate (null) bound may be carried through to $E_\varepsilon^k(\tau)$.

We first integrate around the region Ω_τ and apply Stokes' theorem to obtain

$$E_\varepsilon^k(\tau) - E_\varepsilon^k(0) = \sum_{j=0}^k \int_{\Omega_\tau} t_b \nabla_a^\varepsilon T_{\varepsilon,k}^{ab} \mu^\varepsilon$$

where

$$\begin{aligned} \nabla_a^\varepsilon T_{\varepsilon,k}^{ab} &= e_\varepsilon^{p_1 q_1} \dots e_\varepsilon^{p_{k-1} q_{k-1}} g_\varepsilon^{ac} g_\varepsilon^{bd} \nabla_a^\varepsilon \nabla_c^\varepsilon \nabla_{p_1}^\varepsilon \dots \nabla_{p_{k-1}}^\varepsilon u_\varepsilon \nabla_d^\varepsilon \nabla_{q_1}^\varepsilon \dots \nabla_{q_{k-1}}^\varepsilon u_\varepsilon \\ &\quad + 2e_\varepsilon^{p_1 q_1} \dots e_\varepsilon^{p_{k-1} q_{k-1}} g_\varepsilon^{ab} g_\varepsilon^{cd} \nabla_c^\varepsilon \nabla_{p_1}^\varepsilon \dots \nabla_{p_{k-1}}^\varepsilon u_\varepsilon \nabla_{[a}^\varepsilon \nabla_{d]}^\varepsilon \nabla_{q_1}^\varepsilon \dots \nabla_{q_{k-1}}^\varepsilon u_\varepsilon \\ \nabla_a^\varepsilon T_{\varepsilon,0}^{ab} &= -\nabla_\varepsilon^b u_\varepsilon u_\varepsilon \end{aligned} \quad (11)$$

This expression involves derivatives of order $k+1$, but we may eliminate these substituting the wave equation into it. For the base energy case $k=1$;

$$\begin{aligned} \nabla_a^\varepsilon T_{\varepsilon,1}^{ab} &= f_\varepsilon \nabla_\varepsilon^b u_\varepsilon \\ \nabla_a^\varepsilon T_{\varepsilon,0}^{ab} &= -u_\varepsilon \nabla_\varepsilon^b u_\varepsilon \end{aligned}$$

Therefore

$$E_\varepsilon^1(\tau) = E_\varepsilon^1(0) + \int_{\Omega_\tau} t^a \nabla_a^\varepsilon u_\varepsilon (f_\varepsilon - u_\varepsilon) \mu^\varepsilon$$

On making estimates;

$$\begin{aligned} E_\varepsilon^1(\tau) &\leq E_\varepsilon^1(0) + L_1 \|u_\varepsilon\|_{\Omega_\tau}^1 (\|f_\varepsilon\|_{\Omega_\tau}^0 + \|u_\varepsilon\|_{\Omega_\tau}^0) \\ &\leq E_\varepsilon^1(0) + \frac{1}{2} L_1 (\|f_\varepsilon\|_{\Omega_\tau}^0)^2 + \frac{3}{2} L_1 (\|u_\varepsilon\|_{\Omega_\tau}^1)^2 \end{aligned}$$

where

$$\begin{aligned} \|u_\varepsilon\|_{\Omega}^k &= \left\{ \sum_{\substack{p_1 \dots p_j \\ 0 \leq j \leq k}} \int_{\Omega} |\nabla_{p_1}^\varepsilon \dots \nabla_{p_j}^\varepsilon u_\varepsilon|^2 \mu^\varepsilon \right\}^{1/2} \\ &= \left\{ \int_{\zeta=0}^\tau (\|u_\varepsilon\|_{S_\zeta}^k)^2 d\zeta \right\}^{1/2} \end{aligned}$$

On applying (8) this inequality may be written as

$$E_\varepsilon^1(\tau) \leq E_\varepsilon^1(0) + \frac{1}{2} L_1 (\|f_\varepsilon\|_{\Omega_\tau}^0)^2 + \frac{3}{2} L_1 A'_1 \int_{\zeta=0}^\tau E_\varepsilon^1(\zeta) d\zeta \quad (12)$$

An energy inequality may be also obtained for higher order energies; we must first eliminate the derivatives of order $k+1$ from the expression (11) by using the differentiated wave equation

$$g_\varepsilon^{ab} \nabla_{p_1}^\varepsilon \cdots \nabla_{p_{k-1}}^\varepsilon \nabla_a^\varepsilon \nabla_b^\varepsilon u_\varepsilon = \nabla_{p_1}^\varepsilon \cdots \nabla_{p_{k-1}}^\varepsilon f_\varepsilon$$

In order to substitute this into (11) we must shuffle the covariant derivative indices of u by repeatedly applying the Ricci identities

$$2\nabla_{[a} \nabla_{b]} X_{p_1 p_2 \dots p_k} = R_{ab p_1}{}^c X_{c p_2 \dots p_k} + R_{ab p_2}{}^c X_{p_1 c \dots p_k} + \cdots + R_{ab p_k}{}^c X_{p_1 p_2 \dots c}$$

In this way we find that that

$$g_\varepsilon^{ab} \nabla_a^\varepsilon \nabla_b^\varepsilon \nabla_{p_1}^\varepsilon \cdots \nabla_{p_{k-1}}^\varepsilon u_\varepsilon = \nabla_{p_1}^\varepsilon \cdots \nabla_{p_{k-1}}^\varepsilon f_\varepsilon + \sum_{j=1}^{k-1} \mathcal{R}_{p_1 \dots p_{k-1}}^{(k-1,j)} u_\varepsilon \quad (13)$$

$$2\nabla_{[a} \nabla_{b]} \nabla_{p_1}^\varepsilon \cdots \nabla_{p_{k-1}}^\varepsilon u_\varepsilon = \mathcal{R}_{ab p_1 \dots p_{k-1}}^{(k-1,k-1)} u_\varepsilon \quad (14)$$

where $\mathcal{R}^{(k,j)} u_\varepsilon$ represents a linear combination of contractions of the $(k-j)$ th covariant derivative of the Riemann tensor with the j th covariant derivative of u_ε . This quantity may be bounded as

$$|\mathcal{R}^{(k,j)} u_\varepsilon|^2 \leq \frac{C_{k,j}}{\varepsilon^{2(2+k-j)}} \sum_{q_1 \dots q_j} |\nabla_{q_1}^\varepsilon \cdots \nabla_{q_j}^\varepsilon u_\varepsilon|^2$$

where $C_{k,j}$ is a uniform constant. Now the expressions (13) and (14) may be bounded as follows;

$$\begin{aligned} |g_\varepsilon^{ab} \nabla_a^\varepsilon \nabla_b^\varepsilon \nabla_{p_1}^\varepsilon \cdots \nabla_{p_{k-1}}^\varepsilon u_\varepsilon|^2 &\leq k |\nabla_{p_1}^\varepsilon \cdots \nabla_{p_{k-1}}^\varepsilon f_\varepsilon|^2 + k \sum_{j=1}^{k-1} |\mathcal{R}_{p_1 \dots p_{k-1}}^{(k-1,j)} u_\varepsilon|^2 \\ &\leq C_k \sum_{q_1 \dots q_k} |\nabla_{q_1}^\varepsilon \cdots \nabla_{q_{k-1}}^\varepsilon f_\varepsilon|^2 \\ &\quad + C_k \sum_{\substack{q_1 \dots q_j \\ 1 \leq j \leq k-1}} \frac{1}{\varepsilon^{2(1+k-j)}} |\nabla_{q_1}^\varepsilon \cdots \nabla_{q_j}^\varepsilon u_\varepsilon|^2 \\ |2\nabla_{[a} \nabla_{b]} \nabla_{p_1}^\varepsilon \cdots \nabla_{p_{k-1}}^\varepsilon u_\varepsilon|^2 &\leq \frac{C_k}{\varepsilon^4} \sum_{q_1 \dots q_{k-1}} |\nabla_{q_1}^\varepsilon \cdots \nabla_{q_{k-1}}^\varepsilon u_\varepsilon|^2 \end{aligned}$$

where C_k is a uniform constant. Therefore

$$\begin{aligned} |\nabla_a^\varepsilon T_{\varepsilon,k}^{ab}| &\leq C'_k \left\{ \sum_{p_1 \dots p_k} |\nabla_{p_1}^\varepsilon \cdots \nabla_{p_k}^\varepsilon u_\varepsilon|^2 \right\}^{1/2} \\ &\quad \times \left\{ \sum_{p_1 \dots p_{k-1}} |\nabla_{p_1}^\varepsilon \cdots \nabla_{p_{k-1}}^\varepsilon f_\varepsilon|^2 + \sum_{\substack{p_1 \dots p_j \\ 1 \leq j \leq k-1}} \frac{1}{\varepsilon^{2(1+k-j)}} |\nabla_{p_1}^\varepsilon \cdots \nabla_{p_j}^\varepsilon u_\varepsilon|^2 \right\}^{1/2} \\ &\leq \frac{1}{2} C'_k \left\{ \sum_{p_1 \dots p_k} |\nabla_{p_1}^\varepsilon \cdots \nabla_{p_k}^\varepsilon u_\varepsilon|^2 + \sum_{p_1 \dots p_{k-1}} |\nabla_{p_1}^\varepsilon \cdots \nabla_{p_{k-1}}^\varepsilon f_\varepsilon|^2 + \sum_{\substack{p_1 \dots p_j \\ 1 \leq j \leq k-1}} \frac{1}{\varepsilon^{2(1+k-j)}} |\nabla_{p_1}^\varepsilon \cdots \nabla_{p_j}^\varepsilon u_\varepsilon|^2 \right\} \end{aligned}$$

On integrating this becomes

$$E_\varepsilon^k(\tau) \leq E_\varepsilon^k(0) + C''_k \left\{ (\|u_\varepsilon\|_{\Omega_\tau}^k)^2 + (\|f_\varepsilon\|_{\Omega_\tau}^{k-1})^2 + \sum_{j=1}^{k-1} \frac{1}{\varepsilon^{2(1+k-j)}} (\|u_\varepsilon\|_{\Omega_\tau}^j)^2 \right\}$$

which may be turned into an energy inequality by applying Lemma 1

$$\begin{aligned} E_\varepsilon^k(\tau) &\leq E_\varepsilon^k(0) + C'_k (\|f_\varepsilon\|_{\Omega_\tau}^{k-1})^2 + C''_k \int_{\zeta=0}^\tau E_\varepsilon^k(\zeta) d\zeta \\ &\quad + C'''_k \sum_{j=1}^{k-1} \frac{1}{\varepsilon^{2(2+k-j)}} \int_{\zeta=0}^\tau E_\varepsilon^j(\zeta) d\zeta. \end{aligned} \quad (15)$$

Having obtained this energy inequality we shall prove the following

Lemma 2. Suppose that $\|f_\varepsilon\|_{\Omega_\tau}^0, \dots, \|f_\varepsilon\|_{\Omega_\tau}^{k-1}$ have a null bounds. Then

- (1) if $E_\varepsilon^0(0), \dots, E_\varepsilon^{k-1}(0)$ have moderate bounds then so does $E_\varepsilon^k(\tau)$
- (2) if $E_\varepsilon^0(0), \dots, E_\varepsilon^{k-1}(0)$ have null bounds then so does $E_\varepsilon^k(\tau)$

Proof. For part 1, we proceed by induction (Proof of part 2 is similar). For the case $k = 1$ we may apply Gronwall's Lemma to the energy inequality (12)

$$E_\varepsilon^1(\tau) \leq \left(E_\varepsilon^1(0) + \frac{1}{2} L_1 (\|f_\varepsilon\|_{\Omega_\tau}^0)^2 \right) e^{3L_1 \tau / 2}$$

therefore if for $\varphi \in \mathcal{A}_q$ ($q \geq N$) it is the case that $E_\varepsilon^1(0) = O(\varepsilon^{-N})$ and $\|f_\varepsilon\|_{\Omega_\tau}^0 = O(\varepsilon^{\gamma_q - N})$ it will also be the case that $E_\varepsilon^1(\tau) = O(\varepsilon^{-N})$

Now suppose we have for $\varphi \in \mathcal{A}_q$ ($q \geq N$);

$$\begin{aligned} \|f_\varepsilon\|_{\Omega_\tau}^j &= O(\varepsilon^{\gamma_q - N}) & j &= 0 \dots k-1 \\ E_\varepsilon^j(\tau) &= O(\varepsilon^{-N}) \\ E_\varepsilon^k(0) &= O(\varepsilon^{-N}) \end{aligned}$$

on applying Gronwall's inequality to (15) gives

$$E_\varepsilon^k(\tau) \leq \left\{ E_\varepsilon^k(0) + C'_k (\|f_\varepsilon\|_{\Omega_\tau}^{k-1})^2 + C''_k \sum_{j=1}^k \frac{1}{\varepsilon^{2(2+k-j)}} \int_{\zeta=0}^\tau E_\varepsilon^j(\zeta) d\zeta \right\} e^{C''' \tau}$$

therefore

$$E_\varepsilon^k(\tau) = O(\varepsilon^{-(N+2k)})$$

We now want to apply Lemma 2 to show that if $v_\varepsilon, w_\varepsilon \in \mathcal{E}_M(S)$ then $u_\varepsilon \in \mathcal{E}_M(M)$ and similarly if $v_\varepsilon, w_\varepsilon \in \mathcal{N}(S)$ then $u_\varepsilon \in \mathcal{N}(M)$.

The initial data $(v_\varepsilon, w_\varepsilon)$ may be used to determine the initial energies $E_\varepsilon^k(0)$ by differentiating the Cauchy problem (3) and using the wave equation as follows

$$\begin{aligned} \partial_{\rho_1} \dots \partial_{\rho_k} u_\varepsilon(0, x^\alpha) &= \partial_{\rho_1} \dots \partial_{\rho_k} v_\varepsilon(x^\alpha) \\ \partial_t \partial_{\rho_1} \dots \partial_{\rho_k} u_\varepsilon(0, x^\alpha) &= \partial_{\rho_1} \dots \partial_{\rho_k} w_\varepsilon(x^\alpha) \\ \partial_t^j \partial_{\rho_1} \dots \partial_{\rho_k} u_\varepsilon(0, x^\alpha) &= \partial_{\rho_1} \dots \partial_{\rho_k} \{ g_\varepsilon^{\alpha\beta} \partial_\alpha \partial_\beta \partial_t^j u_\varepsilon + g_\varepsilon^{\alpha\beta} \Gamma_{\alpha\beta}^\gamma \partial_\gamma \partial_t^j u_\varepsilon \}(0, x^\alpha) \\ &\quad - \partial_t^{j-2} \partial_{\rho_1} \dots \partial_{\rho_k} f_\varepsilon(0, x^\alpha) \end{aligned} \tag{16}$$

and then using (6) to define the energy integrals

Equation (16) may be expressed as

$$\begin{aligned} \partial_t^j \partial_{\rho_1} \dots \partial_{\rho_k} u_\varepsilon(0, x^\alpha) &= \sum_{l=1}^{k+2} G_{l\rho_1 \dots \rho_k}^{\sigma_1 \dots \sigma_l}(0, x^\alpha) \partial_{\sigma_1} \dots \partial_{\sigma_l} \partial_t^{j-2} u_\varepsilon(0, x^\alpha) \\ &\quad - \partial_t^{j-2} \partial_{\rho_1} \dots \partial_{\rho_k} f_\varepsilon(0, x^\alpha) \end{aligned}$$

where $G_{l\rho_1 \dots \rho_k}^{\sigma_1 \dots \sigma_l}$ is constructed by sums and products of the metric and its derivatives, hence is an element of $\mathcal{E}_M(S)$. It therefore follows (by an inductive argument) that

- (1) $v_\varepsilon, w_\varepsilon \in \mathcal{E}_M(S)$ and $f_\varepsilon \in \mathcal{N}(M)$ implies that $\partial_t^j \partial_{\rho_1} \dots \partial_{\rho_k} u_\varepsilon(0, x^\alpha) \in \mathcal{E}_M(S)$
- (2) $v_\varepsilon, w_\varepsilon \in \mathcal{N}(S)$ and $f_\varepsilon \in \mathcal{N}(M)$ implies that $\partial_t^j \partial_{\rho_1} \dots \partial_{\rho_k} u_\varepsilon(0, x^\alpha) \in \mathcal{N}(S)$

We therefore have proved the following

Lemma 3. *Suppose that $(v_\varepsilon), (w_\varepsilon) \in \mathcal{E}_M(S)$ and $(f_\varepsilon) \in \mathcal{N}(S)$ then $E_\varepsilon^k(0)$ has a moderate bound, and if in addition that $(v_\varepsilon), (w_\varepsilon) \in \mathcal{N}(S)$ then $E_\varepsilon^k(0)$ has a null bound.*

We also want to translate the bounds for $E_\varepsilon^k(\tau)$, as given by Lemma 2 back to bounds for u_ε and its derivatives. This may be done by applying Lemma 1 in conjunction with the Sobolev embedding theorem (Hawking and Ellis, 1973). Suppose M is a manifold admitting an n dimensional embedded submanifold M' then according to the theorem given $k > 0$, $\exists M_k > 0$ such that $\forall u \in \mathcal{H}^{k+m}(M')$ with $(2m > n)$

$$|\partial_{\rho_1} \dots \partial_{\rho_k} u| \leq M_k \|u\|_{M'}^{k+m} \quad (17)$$

where

$$\|u\|_{M'}^k = \left\{ \sum_{\substack{\rho_1 \dots \rho_j \\ 0 \leq j \leq k}} \int_{M'} |\partial_{\rho_1} \dots \partial_{\rho_j} u|^2 \mu_S \right\}^{1/2} \quad (18)$$

and $\mathcal{H}^k(M')$ is the Sobolev space of functions u existing almost everywhere and for which $\|u\|_{M'}^k < \infty$.

It should be carefully observed that this theorem only gives bounds on the derivatives in the tangential directions to M' and likewise it shows that they may be bounded above by a Sobolev norm that only involves those derivatives.

We first of all apply (17) to u_ε on the submanifold S_τ . This gives us for u_ε

$$|\partial_{\rho_1} \dots \partial_{\rho_k} u_\varepsilon| \leq P_{0,k} \|u_\varepsilon\|_{S_\tau}^{k+m} \leq P_{0,k} \|u_\varepsilon\|_{S_\tau}^{k+m} \quad (2m > 3)$$

Where the second inequality comes from the fact that we may replace the tangential Sobolev norm with the more crude version $\|u\|_{M'}^k$ as its upper bound. However the above result does not put any bounds on any of the time derivatives. We do this by applying (17) to $\partial_t^j u_\varepsilon$;

$$\begin{aligned} |\partial_{\rho_1} \dots \partial_{\rho_k} \partial_t^j u_\varepsilon| &\leq P_{j,k} \|\partial_t^j u_\varepsilon\|_{S_\tau}^{k+m} \\ &\leq P_{j,k} \|u_\varepsilon\|_{S_\tau}^{j+k+m} \end{aligned}$$

Therefore we have shown that $\exists M_k > 0$ such that

$$|\partial_{\rho_1} \dots \partial_{\rho_k} u_\varepsilon| \leq M_k \|u_\varepsilon\|_{S_\tau}^{k+m}, \quad (2m > 3) \quad (19)$$

On combining (19) with Lemma 1 we have proved the following

Lemma 4.

- (1) *If $E_\varepsilon^k(\tau)$ has a moderate bound then $u_\varepsilon \in \mathcal{E}_M(M)$*
- (2) *If $E_\varepsilon^k(\tau)$ has a null bound then $u_\varepsilon \in \mathcal{N}(M)$*

On combining Lemmas 2, 3 and 4 we have proved our main theorem.

Theorem 5.

- (1) *If $v_\varepsilon, w_\varepsilon \in \mathcal{E}_M(S)$ and $f_\varepsilon \in \mathcal{N}(M)$ then $u_\varepsilon \in \mathcal{E}_M(M)$*
- (2) *If $v_\varepsilon, w_\varepsilon \in \mathcal{N}(S)$ and $f_\varepsilon \in \mathcal{N}(M)$ then $u_\varepsilon \in \mathcal{N}(M)$*

We therefore conclude that for metrics of the form (5), a unique solution $U \in \mathcal{G}(M)$ exists to the Cauchy problem (2).

5. THE DISTRIBUTIONAL INTERPRETATION OF THE SOLUTION

It has been established that the solution u_ε may be interpreted as a representative for an element of $\mathcal{G}(M)$. We would like to now discuss whether or not it may be interpreted as a distribution. Thus we want u_ε to be associated to a distribution u ; that is for each $\psi \in \mathcal{D}(M)$, $\exists q \in \mathbb{N}$ such that the limit

$$\langle u, \psi \rangle = \lim_{\varepsilon \rightarrow 0} \int_M u_\varepsilon(\varphi, x) \psi(x) \mu^\varepsilon(x) \quad (20)$$

is well defined and independent of φ .

To do this we consider a representation for u_ε in terms of a Greens function. Assuming that $u_\varepsilon = 0$ for $t < 0$, we may write the Cauchy problem in an ‘inhomogeneous’ form

$$\square^\varepsilon u_\varepsilon^+ = f_\varepsilon^+$$

where

$$\begin{aligned} u_\varepsilon^+(t, x^\alpha) &= H(t)u_\varepsilon(x^\alpha) \\ f_\varepsilon^+(t, x^\alpha) &= H(t)f_\varepsilon(x^\alpha) - \delta'(t)v_\varepsilon(x^\alpha) - \delta(t)w_\varepsilon(x^\alpha) \end{aligned}$$

In this way one may express the solution in terms of a Greens function (see Appendix B)

$$u_\varepsilon^+(x) = \int G_{\varepsilon, x}^+(\xi) f_\varepsilon^+(\xi) \mu^\varepsilon$$

It is therefore the case that if the limit (20) exists it will be given by

$$\langle u, \psi \rangle = \lim_{\varepsilon \rightarrow 0} \int_M u_\varepsilon^+(\varphi, x) \psi(x) \mu^\varepsilon \quad (21)$$

On expressing u_ε^+ in terms of the Greens function and using the self adjoint property of \square together with the fact that G^+ is integrable on $M \times M$, the order of integration may be interchanged giving

$$\begin{aligned} \int_M u_\varepsilon^+(\varphi, x) \psi(x) \mu &= \int_M \left(\int_M G_{\varepsilon, x}^+(\xi) f_\varepsilon^+(\xi) \mu(\xi) \right) \psi(x) \mu^\varepsilon(x) \\ &= \int_{M \times M} G_\varepsilon^+(x, \xi) f_\varepsilon^+(\xi) \psi(x) \mu^\varepsilon(x, \xi) \\ &= \int_{M \times M} G_\varepsilon^-(x, \xi) f_\varepsilon^+(x) \psi(\xi) \mu^\varepsilon(x, \xi) \\ &= \int_M \lambda_\varepsilon(x) f_\varepsilon(x) \mu^\varepsilon(x) \\ &\quad - \int_S (\partial_t \lambda_\varepsilon(0, x^\alpha) v_\varepsilon(x^\alpha) + \lambda_\varepsilon(0, x^\alpha) w_\varepsilon(x^\alpha)) \mu_S^\varepsilon(x^\alpha) \end{aligned} \quad (22)$$

where

$$\lambda_\varepsilon(x) = \int_M G_{\varepsilon, x}^-(\xi) \psi(\xi) \mu^\varepsilon(\xi)$$

Since f_ε is null the first term on the right hand side of (22) will vanish as $\varepsilon \rightarrow 0$ for large enough $q \in \mathbb{N}$, provided that λ_ε admits at most a moderate growth. For the second term to admit a well defined limit, independent of $\varphi \in \mathcal{A}_q$, as $\varepsilon \rightarrow 0$, we require that the limiting function of λ_ε is also well defined and independent of φ . This is because v_ε and w_ε are the embeddings of locally square integrable functions on S . Thus to show that u_ε is associated to a distribution we need to show that λ_ε is sufficiently well behaved. To do this we follow the approach of Bruhat (1962) and use the fact that for $\varepsilon > 0$, $\lambda_\varepsilon(x)$ is a solution of a Volterra type integral equation involving the biscalar K (see Appendix B for details)

$$\lambda_\varepsilon(x) + \frac{1}{2\pi} \int_{C_\varepsilon^-(x)} \square^\varepsilon K_\varepsilon(\xi, x) \lambda_\varepsilon(\xi) \mu_\Gamma^\varepsilon(\xi) = \frac{1}{2\pi} \int_{C_\varepsilon^-(x)} K_\varepsilon(\xi, x) \psi(\xi) \mu_\Gamma^\varepsilon(\xi) \quad (23)$$

where μ_Γ^ε is the volume element induced by μ^ε on $C_\varepsilon^-(x)$. We next examine the limiting behaviour of the terms in (23). We first consider the integral on the left hand side. In Appendix B it is shown that in normal coordinates based at ξ

$$\square^\varepsilon K_\varepsilon(\xi, x) = -\frac{1}{6} K_\varepsilon(\xi, x) R_\varepsilon(\xi) + O(\xi^2) \quad (24)$$

On the other hand using the results of Clarke et al (1996)

$$\lim_{\varepsilon \rightarrow 0} \int R_\varepsilon(\xi) \Psi(\xi) \mu(\xi) = 4\pi(1 - A) \int \Psi(\xi^0, 0, 0, \xi^3) d\xi^0 d\xi^3 \quad (25)$$

So that provided λ_0 is well defined then

$$\lim_{\varepsilon \rightarrow 0} \int_{C_\varepsilon^-(x)} \square^\varepsilon K_\varepsilon(\xi, x) \lambda_\varepsilon(\xi) \mu_\Gamma^\varepsilon(\xi) = -\frac{2}{3}(1-A)\pi \int_{\Lambda^-(x)} \lambda_0(\xi) \mu_{\Lambda^-}(\xi) \quad (26)$$

where $\Lambda^-(x)$ is the limit of the intersection of $C_\varepsilon^-(x)$ with the axis Λ and μ_{Λ^-} is the volume form induced on $\Lambda^-(x)$. It is shown in Appendix C that these quantities are well defined.

The integral of the right hand side of (23) is more straightforward since for $x, \xi \notin \Lambda$, $K_\varepsilon(\xi, x) \rightarrow 1$ as $\varepsilon \rightarrow 0$. Also we show in Appendix C that $C_\varepsilon^-(x)$ tends to a well defined limit $C_0^-(x)$. However some care must be taken in interpreting $C_0^-(x)$. One cannot simply take this to be the past null cone of x in the conical space-time with the axis removed. This is because deleting the axis results in a tear in the null cone and destroys the S^2 topology. However (in the case of a rotationally symmetric smoothing) it is shown in Appendix C that $C_0^-(x) = \lim_{\varepsilon \rightarrow 0} C_\varepsilon^-(x)$ is well defined and does have S^2 topology the missing piece being generated by geodesics which pass within $O(\varepsilon)$ of the axis Λ , and which hit the axis in the limit. One can also show that the corresponding volume form $\mu_\Gamma^\varepsilon(\xi)$ also has a well defined limit for $\xi \notin \Lambda$ given by the volume form induced by the conical metric on $C_0^-(x)$. Furthermore the value on the axis is bounded so that we need not include the contribution from the integral over the axis. Hence provided that λ_0 exists we have

$$\lim_{\varepsilon \rightarrow 0} \int_{C_\varepsilon^-(x)} K_\varepsilon(\xi, x) \psi(\xi) \mu_\Gamma^\varepsilon(\xi) = \int_{\hat{C}_0^-(x)} \psi(\xi) \mu_\Gamma^0(\xi) \quad (27)$$

where $\hat{C}_0^-(x) = C_0^-(x) \setminus \Lambda$.

On the basis of (26) and (27) we now *define* $\lambda(x)$, to be the solution of

$$\lambda(x) - \frac{1}{3}(1-A) \int_{\Lambda^-(x)} \lambda_0(\xi) \mu_\Lambda(\xi) = \frac{1}{2\pi} \int_{\hat{C}_0^-(x)} \psi(\xi) \mu_\Gamma^0(\xi) \quad (28)$$

Note that this gives λ in terms of ψ on $\hat{C}_0^-(x)$ and λ on $C_0^-(x) \cap \Lambda$. For a point x that lies on the axis $\Lambda^-(x)$ degenerates into a pair of null lines and μ_Λ vanishes so that the integral on the left hand side of (28) vanishes and we have

$$\lambda(x) = \frac{1}{2\pi} \int_{\hat{C}_0^-(x)} \psi(\xi) \mu_\Gamma^0(\xi), \quad x \in \Lambda \quad (29)$$

where for a point x on the axis, the past null cone $\hat{C}_0^-(x)$ is given in quasi-Cartesian coordinates by the usual Minkowskian formula. Equation (29) gives an expression for $\lambda \in \Lambda$ simply in terms of an integral involving ψ , so we may substitute back for λ in the left hand integral of (28) to obtain an integral for λ in terms of ψ which is valid for all x .

$$\lambda(x) = \frac{1}{2\pi} \int_{\xi \in C_0^-(x)} \psi(\xi) \mu_\Gamma^0(\xi) + \frac{1}{6\pi}(1-A) \int_{\xi \in \Lambda^-(x)} \left(\int_{\eta \in C_0^-(\xi)} \psi(\eta) \mu_\Gamma^0(\eta) \right) \mu_\Lambda(\xi) \quad (30)$$

Thus a solution to (28) is given by the above integral (30). Clearly this is a well defined quantity which does not depend on the choice of the smoothing kernel φ . It remains to show that λ_ε does indeed tend to λ as $\varepsilon \rightarrow 0$. To do this we let $\rho_\varepsilon(x) = \lambda(x) - \lambda_\varepsilon(x)$ then subtracting (28) from (23) we find that ρ_ε satisfies the integral equation

$$\begin{aligned} \rho_\varepsilon(x) + \frac{1}{2\pi} \int_{C_\varepsilon^-(x)} \square^\varepsilon K_\varepsilon(\xi, x) \rho_\varepsilon(\xi) \mu_\Gamma^\varepsilon(\xi) &= \frac{1}{2\pi} \int_{C_\varepsilon^-(x)} \square^\varepsilon K_\varepsilon(\xi, x) \lambda(\xi) \mu_\Gamma^\varepsilon(\xi) + \frac{1}{3}(1-A) \int_{\Lambda^-(x)} \lambda_0(\xi) \mu_\Lambda(\xi) \\ &\quad + \frac{1}{2\pi} \int_{\hat{C}_0^-(x)} \psi(\xi) \mu_\Gamma^0(\xi) - \frac{1}{2\pi} \int_{C_\varepsilon^-(x)} \psi(\xi) \mu_\Gamma^\varepsilon(\xi) \end{aligned} \quad (31)$$

where λ is given by (30).

Since we know from (30) that λ is bounded then (24) and (25) show that

$$\frac{1}{2\pi} \int_{C_\varepsilon^-(x)} \square^\varepsilon K_\varepsilon(\xi, x) \lambda(\xi) \mu_\Gamma^\varepsilon(\xi) + \frac{1}{3}(1-A) \int_{\Lambda^-(x)} \lambda_0(\xi) \mu_\Lambda(\xi) = O(\varepsilon)$$

Also from Appendix C we know that $C_\varepsilon^-(x) \rightarrow C_0^-(x)$ as $\varepsilon \rightarrow 0$, and that $\mu_\Gamma^\varepsilon \rightarrow \mu_\Gamma^0$, except possibly on Λ (which is of measure zero) so that

$$\frac{1}{2\pi} \int_{\hat{C}_0^-(x)} \psi(\xi) \mu_\Gamma^0(\xi) - \frac{1}{2\pi} \int_{C_\varepsilon^-(x)} \psi(\xi) \mu_\Gamma^\varepsilon(\xi) = O(\varepsilon)$$

Hence $\rho_\varepsilon(x)$ satisfies the integral equation

$$\rho_\varepsilon(x) + \frac{1}{2\pi} \int_{C_\varepsilon^-(x)} \square^\varepsilon K_\varepsilon(\xi, x) \rho_\varepsilon(\xi) \mu_\Gamma^\varepsilon(\xi) = h_\varepsilon(x) \quad (32)$$

where $h_\varepsilon(x)$ denotes the function on the right hand side of (31) and which tends to zero as $\varepsilon \rightarrow 0$. We may obtain a solution to this equation by iteration and find that $\rho_\varepsilon(x)$ also tends to zero as $\varepsilon \rightarrow 0$.

Thus $\lambda_\varepsilon(x)$ tends to $\lambda(x)$ given by (30) as $\varepsilon \rightarrow 0$ and inserting this into (22) and using (21) we see that for $\varphi \in \mathcal{A}_q$ for q sufficiently large

$$\lim_{\varepsilon \rightarrow 0} \langle u_\varepsilon, \psi \rangle = - \int_{S \setminus \Lambda} (\partial_t \lambda(0, x^\alpha) v(x^\alpha) + \lambda(0, x^\alpha) w(x^\alpha)) \mu_S^0(x^\alpha) \quad (33)$$

where μ_S^0 is the volume form induced by the conical metric on $S \setminus \Lambda$ and $\lambda(t, x^\alpha)$ is given by (30). Thus U is associated to a distribution u defined by (30) and the right hand side of (33).

It is worth remarking that even though $M \setminus \Lambda$ is locally flat the Greens function for the cone is not sharp due to the second term in (30) (which vanishes in Minkowski space when $A = 1$). The solution at x depends not just on the initial data on $C_0^-(x) \cap S$ but also on points in $C_0^-(\xi) \cap S$ with $\xi \in C_0^-(x) \cap \Lambda$. i.e. as well as the sharp term in the Greens function there is an extra term which involves scattering off the axis.

6. CONCLUSION

In an earlier paper (Clarke et al 1996) we showed that it was possible to give a distributional interpretation to the curvature of a conical space-time by using Colombeau's theory of non-linear generalised functions. In this paper we have established that such conical singularities do not disrupt the Cauchy development of test fields on this fixed background. Although it is not feasible to undertake a full non-linear analysis and show that such singularities do not disrupt the Cauchy development of Einstein's equations, the higher order energy estimates obtained in Section 4 indicate that the back reaction is not likely to radically affect the nature of the singularity. These results taken together therefore show that it is reasonable to interpret conical space-times not only as distributional geometries but also as distributional solutions of Einstein's equations.

The concept of generalised hyperbolicity considered in this paper (\mathcal{G} -hyperbolicity) is different from that adopted by Clarke (1998) when considering curve-integrable space-times. However the curve integrable condition is primarily used to construct a geodesic congruence with tangent vector having bounded covariant derivative, and such a congruence may be explicitly constructed in conical space-times even though they fail to be curve-integrable. In a future paper we will examine the relationship between these two conditions.

Finally it is worth remarking that the limiting solution that one obtains to the wave equation is more interesting than one might at first expect. It contains a term due to the delta-function curvature on the axis and therefore has tail terms even though the conical space-time $(M \setminus \Lambda, g_0)$ is locally flat. It also contains a term which arises from parts of the null cone which in the limit pass through the axis and 'mends the tear'. A naive approach in which one looked at the solution in Minkowski space in cylindrical polar coordinates and then rescaled the angular coordinate would not produce the correct answer.

APPENDIX A. RESTRICTIONS IN THE COLOMBEAU ALGEBRA

When dealing with initial value or boundary value problems in the context of Colombeau algebras one is interested in solving some differential equation on \mathbb{R}^n , but giving data on some lower dimensional subspace $S \equiv \mathbb{R}^m$. For example when solving the initial value problem for the wave equation in $\mathcal{G}(\mathbb{R}^4)$

$$\begin{aligned} \square U &= 0 \\ U|_S &= V \\ \partial_t U|_S &= W \end{aligned}$$

where $S = \{(t, x, y, z) \in \mathbb{R}^4 : t = 0\} \equiv \mathbb{R}^3$, we need to regard $U \in \mathcal{G}(\mathbb{R}^4)$ but $U|_S \in \mathcal{G}(\mathbb{R}^3)$. One way of doing this is to require that elements $\varphi^{(n)} \in \mathcal{A}(\mathbb{R}^n)$ are products of one dimensional kernels $\varphi \in \mathcal{A}(\mathbb{R})$ so that

$$\varphi^{(n)}(x^1, \dots, x^n) = \varphi(x^1) \cdots \varphi(x^n)$$

If $U \in \mathcal{G}(\mathbb{R}^4)$ then we may define $U|_S \in \mathcal{G}(\mathbb{R}^3)$ by

$$U|_S(\varphi^{(3)}, (x, y, z)) = U(\varphi^{(4)}, (0, x, y, z)) \quad (\text{A.1})$$

Unfortunately we often want to restrict $\mathcal{A}_q(\mathbb{R}^4)$ to some subset which is invariant under the action of some symmetry group (for example rotations about the z -axis) and it is not usually possible to achieve this with kernels taking the form (A.1). In this appendix we will therefore describe a way of defining restrictions which allows one to work with kernels with a specified symmetry. We will illustrate the approach by restricting $\mathcal{G}(\mathbb{R}^4)$ to a three dimensional subspace, but the construction can be readily generalised to any linear subspace of \mathbb{R}^n .

Let $\tilde{f} \in \mathcal{E}_M(\mathbb{R}^4)$, then we define

$$\begin{aligned} \tilde{f}|_S : \mathcal{A}_q(\mathbb{R}^3) \times \mathbb{R}^3 &\rightarrow \mathbb{R} \\ \text{by } \tilde{f}|_S(\varphi, (x, y, z)) &= \tilde{f}(\hat{\varphi}, (0, x, y, z)) \end{aligned}$$

where

$$\hat{\varphi}(t, x, y, z) = \frac{1}{3} \varphi(x, y, z) \int_{\mathbb{R}^2} (\varphi(t, u, v) + \varphi(v, t, u) + \varphi(u, v, t)) du dv$$

The first important point to note is that if $\varphi \in \mathcal{A}(\mathbb{R}^3)$ is invariant under some symmetry group (such as rotations about the z -axis) then $\hat{\varphi} \in \mathcal{A}(\mathbb{R}^4)$ is invariant under the same symmetry. The second important feature of the construction is given by the following Lemma.

Lemma 6.

- (1) $\varphi \in \mathcal{A}_0(\mathbb{R}^3) \implies \hat{\varphi} \in \mathcal{A}_0(\mathbb{R}^4)$
- (2) $\varphi \in \mathcal{A}_q(\mathbb{R}^3) \implies \hat{\varphi} \in \mathcal{A}_q(\mathbb{R}^4)$

Proof. For (1) we note

$$\begin{aligned} \int_{\mathbb{R}^4} \hat{\varphi}(t, x, y, z) dt dx dy dz &= \frac{1}{3} \int_{\mathbb{R}^3} \varphi(x, y, z) dx dy dz \int_{\mathbb{R}^3} (\varphi(t, u, v) + \varphi(v, t, u) + \varphi(u, v, t)) du dv dt \\ &= 1 \end{aligned}$$

For (2) we let $\varphi \in \mathcal{A}_q(\mathbb{R}^3)$ with $q \geq 1$ and let $a, b, c, d \in \mathbb{N}$ be such that $1 \leq a + b + c + d \leq q$. Then

$$\begin{aligned} \int_{\mathbb{R}^4} \hat{\varphi}(t, x, y, z) t^a x^b y^c z^d dt dx dy dz \\ = \frac{1}{3} \int_{\mathbb{R}^3} \varphi(x, y, z) x^b y^c z^d dx dy dz \int_{\mathbb{R}^3} (\varphi(t, u, v) + \varphi(v, t, u) + \varphi(u, v, t)) t^a du dv dt \end{aligned}$$

If $a \geq 1$, the second integral on the right vanishes, while if $a = 0$ the first integral on the right vanishes. In either case the left hand side vanishes and $\hat{\varphi} \in \mathcal{A}_q(\mathbb{R}^4)$.

Proposition 7.

- (1) $\tilde{f} \in \mathcal{E}_M(\mathbb{R}^4) \implies \tilde{f}|_S \in \mathcal{E}_M(\mathbb{R}^3)$
- (2) $\tilde{f} \in \mathcal{N}(\mathbb{R}^4) \implies \tilde{f}|_S \in \mathcal{N}(\mathbb{R}^3)$

Proof. The proof of (1) and (2) follows directly from the definition of moderate and null together with Lemma 6.

Corollary 8. Let $F \in \mathcal{G}(\mathbb{R}^4)$ have representative $\tilde{f} \in \mathcal{E}_M(\mathbb{R}^4)$, then $F|_S \in \mathcal{G}(\mathbb{R}^3)$ may be defined by $F|_S = [\tilde{f}|_S]$.

We have therefore shown how it is possible to restrict elements of $\mathcal{G}(\mathbb{R}^4)$ to some subspace S while maintaining the required symmetry. The following result shows that for smooth functions this notion of restriction commutes with the canonical embedding $\iota_n : C^\infty(\mathbb{R}^n) \rightarrow \mathcal{G}(\mathbb{R}^n)$.

Proposition 9. *Let $f \in C^\infty(\mathbb{R}^4)$ then*

$$(\iota_4 f)|_S = \iota_3(f|_S)$$

Proof.

$$\begin{aligned} \widetilde{(f|_S)}_\varepsilon &= \int_{\mathbb{R}^3} f(0, x + \varepsilon x', y + \varepsilon y', z + \varepsilon z') \varphi(x', y', z') dx' dy' dz' \\ (\tilde{f}_\varepsilon)|_S &= \frac{1}{3} \int_{\mathbb{R}^6} f(\varepsilon t', x + \varepsilon x', y + \varepsilon y', z + \varepsilon z') \varphi(x', y', z') \\ &\quad \times (\varphi(t', u, v) + \varphi(v, t', u) + \varphi(u, v, t')) du dv dt' dx' dy' dz' \end{aligned}$$

Then by looking at the Taylor expansion of $\widetilde{(f|_S)}_\varepsilon - (\tilde{f}_\varepsilon)|_S$ one can verify that the difference is null.

APPENDIX B. THE GREENS FUNCTION FOR THE WAVE OPERATOR \square .

In this appendix we summarise some results of Friedlander (1975) and Bruhat (1962). Let $x \in M$. A distribution may be defined on $D^-(x)$ as follows; let

$$S_\varepsilon^-(x) = \{\xi \in D^-(x) | \Gamma(\xi, x) = \varepsilon\}$$

where $\Gamma(\xi, x)$ denotes the square distance between x and ξ in M . We then define a distribution $\delta^-(\Gamma(\xi, x) - \varepsilon)$ by

$$\langle \delta^-(\Gamma(\xi, x) - \varepsilon), \varphi(\xi) \rangle = \int_{S_\varepsilon^-(x)} \varphi(\xi) \mu_\Gamma(\xi)$$

Consequently we may define a limiting distribution $\delta^-(\xi, x)$ by

$$\langle \delta^-(\xi, x), \varphi(\xi) \rangle = \lim_{\varepsilon \rightarrow 0^+} \int_{S_\varepsilon^-(x)} \varphi(\xi) \mu_\Gamma(\xi)$$

It may be shown, for the wave operator \square (Friedlander 1975, Theorem 4.2.1), that there exists a function $K \in C^\infty \times C^\infty$ (the biscalar) such that

$$\square(K(\xi, x)\delta^-(\xi, x)) = (\square K(\xi, x))\delta^-(\xi, x) + 2\pi\delta(\xi - x) \quad (\text{B.1})$$

and moreover K can be chosen so that

$$\begin{aligned} 2\nabla^a \Gamma \nabla_a K + (\square \Gamma - 8)K &= 0 \\ K(x, x) &= 1 \end{aligned} \quad (\text{B.2})$$

On integrating (B.1) we obtain

$$\int_M \square(K(\xi, x)\delta^-(\xi, x))\varphi(\xi)\mu(\xi) = \int_M (\square K(\xi, x))\delta^-(\xi, x)\varphi(\xi)\mu + 2\pi\varphi(x)$$

which may be rearranged, by integrating by parts and using the fact that φ has compact support, to obtain

$$\varphi(x) = \frac{1}{2\pi} \int_{C^-(x)} K(\xi, x) \square \varphi(\xi) \mu_\Gamma(\xi) - \frac{1}{2\pi} \int_{C^-(x)} \square K(\xi, x) \varphi(\xi) \mu_\Gamma(\xi)$$

This equation may be regarded as a Volterra type integral equation for φ given $\psi \in \mathcal{D}$ ($\square\varphi$ in this case)

$$\varphi(x) + \frac{1}{2\pi} \int_{C^-(x)} \square K(\xi, x) \varphi(\xi) \mu_\Gamma(\xi) = \frac{1}{2\pi} \int_{C^-(x)} K(\xi, x) \psi(\xi) \mu_\Gamma(\xi)$$

This equation naturally defines a distribution $G_x^- : \mathcal{D} \rightarrow \mathbb{R}$ given by $\langle G_x^-, \psi \rangle = \varphi(x)$ (Bruhat, 1962). The distribution G_x^- is often known as the past Greens function. A future Greens function G_x^+ may be defined by an analogous procedure, replacing the past null cone $C^-(x)$ with the future null cone $C^+(x)$ in all the definitions above. In fact we may define a distributions on $\mathcal{D} \otimes \mathcal{D}$ by

$$\langle G^\pm(x, \xi), \varphi(x)\psi(\xi) \rangle = \langle \langle E_x^\pm, \psi \rangle, \varphi(x) \rangle$$

The fact that \square is self adjoint implies that $G^+(x, \xi) = G^-(\xi, x)$.

Calculation of K and $\square K$ for the wave operator \square .

We shall first consider $K(x, y)$ in normal coordinates based at x . In normal coordinates we have

$$\begin{aligned} g_{ab}(y)y^b &\stackrel{*}{=} g_{ab}(x)y^b \\ \Gamma(x, y) &\stackrel{*}{=} g_{ab}(x)y^a y^b \end{aligned}$$

This implies that

$$\nabla^a \Gamma \stackrel{*}{=} 2y^a$$

and so

$$\begin{aligned} \square \Gamma &\stackrel{*}{=} |g|^{-1/2} \partial_a (|g|^{1/2} \nabla^a \Gamma) \\ &\stackrel{*}{=} 2|g|^{-1/2} \partial_a (|g|^{1/2} y^a) \\ &\stackrel{*}{=} 8 + y^a \partial_a (\log |g|) \end{aligned}$$

Substituting in (B.2) gives

$$4y^a \partial_a K + y^a \partial_a (\log |g|) K \stackrel{*}{=} 0$$

which implies

$$y^a \partial_a (|g|^{1/4}) \stackrel{*}{=} 0$$

This means that $|g(y)|^{1/4} K(x, y)$ is constant along the geodesic connecting x and y . Therefore

$$\begin{aligned} |g(y)|^{1/4} K(x, y) &\stackrel{*}{=} |g(x)|^{1/4} K(x, x) \\ &\stackrel{*}{=} |g(x)|^{1/4} \end{aligned}$$

Therefore

$$K(x, y) \stackrel{*}{=} \frac{|g(x)|^{1/4}}{|g(y)|^{1/4}} \tag{B.3}$$

We now consider $K(x, y)$ in a more general coordinate system. We shall use primed coordinates to denote the normal coordinate system. Let

$$J(y) = \det \left(\frac{\partial y^a}{\partial y'^b} \right)$$

Then

$$\begin{aligned} K(x, y) &= \frac{|g'(x)|^{1/4}}{|g'(y)|^{1/4}} \\ &= \frac{|g(x)|^{1/4} J(x)^{-1/2}}{|g(y)|^{1/4} J(y)^{-1/2}} \\ &= \frac{|g(x)|^{1/4}}{|g(y)|^{1/4}} J(y)^{1/2} \end{aligned}$$

But

$$\frac{\partial \Gamma}{\partial x^a}(x, y) = -2g_{ab}(x)y'^b$$

So

$$\frac{\partial^2 \Gamma}{\partial y^a \partial x^b}(x, y) = -2g_{bc}(x) \frac{\partial y'^c}{\partial y^a}$$

implying that

$$\det \left(\frac{\partial^2 \Gamma}{\partial y^a \partial x^b}(x, y) \right) = -2g(x)J(y)^{-1}$$

Therefore

$$K(x, y) = \frac{\left| \det\left(\frac{\partial^2 \Gamma}{\partial y^a \partial x^b}\right) \right|^{1/2}}{4|g(x)|^{1/4}|g(y)|^{1/4}}$$

We shall now consider $\square K(x, y)$ in normal coordinates based at x . Applying \square to (B.3)

$$\begin{aligned} \square K &= |g|^{-1/2} \partial_a (g^{ab} |g|^{1/2} \partial_b K) \\ &= |g|^{-1/4} \partial_a (g^{ab} |g|^{1/2} \partial_b (|g|^{-1/4})) \\ &= -\frac{1}{4} K(x, y) \gamma(y) \end{aligned}$$

where

$$\gamma(y) \stackrel{*}{=} (g^{ab} g^{cd} g_{cd,a})_b + \frac{1}{4} g^{ab} g^{cd} g^{ef} g_{cd,a} g_{ef,b}$$

In normal coordinates at x it may be shown that

$$\begin{aligned} g_{ab,c}(x) &\stackrel{*}{=} 0 \\ R(x) &\stackrel{*}{=} \frac{2}{3} g^{ab}(x) g^{cd}(x) g_{cd,ab}(x) \end{aligned}$$

so

$$\gamma(x) \stackrel{*}{=} \frac{2}{3} R(x)$$

We may now expand $g_{ab}(y)$ in powers of y about x ;

$$g_{ab}(y) \stackrel{*}{=} g_{ab}(x) + \frac{1}{3} R_{acbd}(x) y^c y^d + O(y^3)$$

this implies

$$\gamma(y) \stackrel{*}{=} \frac{2}{3} R(x) + \frac{1}{3} (\nabla_a R(x) + 2 \nabla^b R_{ab}(x)) y^a + O(y^2)$$

and on applying the Bianchi identities,

$$\gamma(y) - \frac{2}{3} R(x) \stackrel{*}{=} O(y^2)$$

Therefore

$$\square K(x, y) \stackrel{*}{=} -\frac{1}{6} K(x, y) R(y) + O(y^2) \tag{B.4}$$

a result obtained by McLenaghan (1969).

APPENDIX C: NULL GEODESICS IN A CONICAL SPACE-TIME.

In this appendix we examine the limit of $C_\varepsilon^-(x)$, the past null cone of x in (M, g_ε) as $\varepsilon \rightarrow 0$. To do this we first examine the past null cone of points in $(M \setminus \Lambda, g_0)$, the conical space-time with the axis removed. Although it is not hard to derive the equation of the null cone by working in the Minkowskian covering space and then making the appropriate identifications, we will derive it using a method that can also be applied to $(M \setminus \Lambda, g_\varepsilon)$.

If we work in polar coordinates the geodesics may be found using the Lagrangian

$$L = \frac{1}{2} \{-\dot{t}^2 + \dot{r}^2 + A^2 r^2 \dot{\phi}^2 + \dot{z}^2\}$$

where a dot indicates the derivative with respect to some affine parameter. The geodesic equations show that we may take t as an affine parameter. In view of the rotational and translational symmetries of the metric, without loss of generality we may consider the null cone of a point P with coordinates $(0, a, 0, 0)$. We therefore consider geodesics which, when $t = 0$, satisfy the initial conditions:

$$\begin{aligned} r &= a & \phi &= 0 & z &= 0 \\ \dot{r} &= -\sin \gamma \cos \beta & \dot{\phi} &= \frac{1}{Aa} \sin \beta \sin \gamma & \dot{z} &= \cos \gamma \end{aligned}$$

where β and γ are polar angles which parameterise the S^2 celestial sphere of null directions at P (with $\beta = 0$ corresponding to a geodesic which hits the axis).

Then the z equation gives

$$z = t \cos \gamma$$

The null geodesics may then be found by solving the equations

$$\begin{aligned} A^2 r^2 \dot{\phi} &= J \\ \dot{r}^2 + A^2 r^2 \dot{\phi}^2 &= E^2 \end{aligned}$$

where

$$\begin{aligned} J &= Aa \sin \beta \sin \gamma \\ E^2 &= \sin^2 \gamma \end{aligned}$$

Substituting for $\dot{\phi}$ this gives

$$\frac{dr}{dt} = \pm \frac{\sin \gamma (r^2 - a^2 \sin^2 \beta)^{1/2}}{r}$$

and hence

$$t = - \int \frac{r dr}{\sin \gamma (r^2 - a^2 \sin^2 \beta)^{1/2}}$$

where the minus sign is taken for the past null cone. Performing the integration and using the fact that $r = a$ when $t = 0$ gives

$$t = - \frac{(r^2 - a^2 \sin^2 \beta)^{1/2}}{\sin \gamma} + \frac{a \cos \beta}{\sin \gamma}$$

Solving for r enables one to find $r(t)$ where

$$r^2(t) = t^2 \sin^2 \gamma - 2at \cos \beta \sin \gamma + a^2 \quad (\text{C.1})$$

We now substitute for r in the ϕ equation to obtain

$$\frac{d\phi}{dt} = \frac{a \sin \beta \sin \gamma}{A(t^2 \sin^2 \gamma - 2at \cos \beta \sin \gamma + a^2)}$$

Integrating and putting in the initial condition that $\phi = 0$ when $t = 0$ gives

$$A\phi = -\beta \pm \cot^{-1} \left(\frac{t \sin \gamma - a \cos \beta}{a \sin \beta} \right) \quad (\text{C.2})$$

We have now obtained $r(t)$, $\phi(t)$ and $z(t)$ and one can now eliminate β and γ and show that t , r , ϕ and z satisfy the constraint

$$a^2 + r^2 - 2ar \cos(A\phi) + z^2 - t^2 = 0 \quad (\text{C.3})$$

The points on the past null cone of $(0, a, 0, 0)$ are therefore the points where $t < 0$ and (C.3) is satisfied.

We now turn to considering the null cone in the regularised space-time (M, g_ε) . If we use a rotationally symmetric smoothing kernel, as described in Appendix A, we may write the metric in the form

$$ds^2 = -dt^2 + P_\varepsilon^2(r) dr^2 + r^2 Q_\varepsilon^2(r) d\phi^2 + dz^2$$

where $P_\varepsilon(r) = P(r/\varepsilon)$, $Q_\varepsilon(r) = Q(r/\varepsilon)$ and P and Q are given by

$$\begin{aligned} P(r)^2 &= \frac{1}{2}(1 + A^2) + \frac{1}{2}(1 - A^2) 2\pi \int_0^r \left(1 - \frac{\rho^2}{r^2}\right) \varphi(\rho) \rho d\rho \\ Q(r)^2 &= \frac{1}{2}(1 + A^2) - \frac{1}{2}(1 - A^2) 2\pi \int_0^r \left(1 - \frac{\rho^2}{r^2}\right) \varphi(\rho) \rho d\rho \end{aligned}$$

Here φ is a smooth function with compact support such that

$$\begin{aligned} 2\pi \int \varphi(r) r dr &= 1 \\ 2\pi \int \varphi(r) r^3 dr &= 0 \end{aligned}$$

so that the metric (when regarded in Cartesian coordinates as a function of φ) may be interpreted as an element of Colombeau's generalised function algebra $\mathcal{G}(\mathbb{R}^4)$.

In this space-time the geodesics may be found by considering the Lagrangian

$$L_\varepsilon = \frac{1}{2}\{-\dot{t}^2 + P_\varepsilon^2 \dot{r}^2 + r^2 Q_\varepsilon^2 \dot{\phi}^2 + \dot{z}^2\}$$

and we see that as before we may take t as an affine parameter. We again consider the null cone of $(0, a, 0, 0)$, but this time take the initial conditions:

$$\begin{aligned} r &= a & \phi &= 0 & z &= 0 \\ \dot{r} &= -\frac{1}{P_\varepsilon(a)} \sin \gamma \cos \beta & \dot{\phi} &= \frac{1}{a Q_\varepsilon(a)} \sin \beta \sin \gamma & \dot{z} &= \cos \gamma \end{aligned}$$

We again have $z(t) = t \cos \gamma$, and the r and t equations are now

$$\begin{aligned} Q_\varepsilon^2(r) r^2 \dot{\phi} &= J_\varepsilon \\ P_\varepsilon^2(r) \dot{r}^2 + r^2 Q_\varepsilon^2(r) \dot{\phi}^2 &= E_\varepsilon^2 \end{aligned}$$

where

$$\begin{aligned} J_\varepsilon &= a Q_\varepsilon(a) \sin \beta \sin \gamma \\ E_\varepsilon^2 &= \sin^2 \gamma \end{aligned}$$

Substituting for $\dot{\phi}$ this gives

$$\frac{dr}{dt} = \pm \frac{\sin \gamma (Q_\varepsilon^2(r) r^2 - a^2 Q_\varepsilon^2(a) \sin^2 \beta)^{1/2}}{P_\varepsilon(r) Q_\varepsilon(r) r}$$

and hence

$$t = - \int \frac{P_\varepsilon(r) r dr}{\sin \gamma (r^2 - a^2 S_\varepsilon(r) \sin^2 \beta)^{1/2}}$$

where $S_\varepsilon(r) = Q_\varepsilon(a)/Q_\varepsilon(r)$.

We now let

$$R_0 = \sup\{r : |\varphi(r)| > 0\}$$

and for the moment restrict attention to geodesics for which $r_{\min} > \varepsilon R_0$. Then for $\varphi \in \mathcal{A}_q$ and $r > \varepsilon$ we have

$$\begin{aligned} P_\varepsilon &= 1 + O\left(\frac{\varepsilon^{q+1}}{r^{q+1}}\right) R_0 \\ Q_\varepsilon &= A + O\left(\frac{\varepsilon^{q+1}}{r^{q+1}}\right) R_0 \end{aligned}$$

These estimates allow us to deduce that

$$t = -\frac{(r^2 - a^2 \sin^2 \beta)^{1/2}}{\sin \gamma} + \frac{a \cos \beta}{\sin \gamma} + g_\varepsilon(r)$$

where $g_\varepsilon(r) \rightarrow 0$ uniformly as $\varepsilon \rightarrow 0$, and hence that

$$r_\varepsilon(t) = r(t) + h_\varepsilon(t) \tag{C.4}$$

where $r(t)$ is given by (C.1) and $h_\varepsilon(t) \rightarrow 0$ uniformly as $\varepsilon \rightarrow 0$.

We now consider the ϕ equation

$$\frac{d\phi}{dt} = \frac{J_\varepsilon}{Q_\varepsilon^2(r) r^2}$$

so that

$$Q_\varepsilon(a) \phi = \int \frac{a S_\varepsilon^2(r) \sin \beta \sin \gamma dt}{t^2 \sin^2 \gamma - 2at \cos \beta \sin \gamma + a^2 + h_\varepsilon(t)}$$

Using the initial conditions for ϕ and fact that $r_\varepsilon(t) > \varepsilon R_0$ together with the estimates for P_ε and Q_ε we may deduce that

$$\phi_\varepsilon(t) = \phi(t) + k_\varepsilon(t) \quad (\text{C.5})$$

where $\phi(t)$ is given by (C.2) and $k_\varepsilon(t) \rightarrow 0$ uniformly as $\varepsilon \rightarrow 0$.

If we now substitute (C.4) and (C.5) into the constraint equation (C.3) for the null cone in a conical space-time we find we get an equation of the form

$$a^2 + r^2 - 2ar \cos(A\phi) + z^2 - t^2 = m_\varepsilon(t, r, \phi, z)$$

where $m_\varepsilon \rightarrow 0$ uniformly as $\varepsilon \rightarrow 0$.

We also note that as $\varepsilon \rightarrow 0$, the excluded geodesics are those which hit the axis. So that if we denote by $\hat{C}^-(x)$ the past null cone of x excluding those points which lie on geodesics which hit the axis, then the above result implies that $\hat{C}_\varepsilon^-(x) \rightarrow \hat{C}_0^-(x)$ as $\varepsilon \rightarrow 0$.

The estimates for P_ε and Q_ε also show that the volume form μ^ε defined by g_ε tends uniformly to a well defined limit μ^0 as $\varepsilon \rightarrow 0$ (at least for $x \notin \Lambda$). So that if $\psi \in \mathcal{D}(M)$ then

$$\lim_{\varepsilon \rightarrow 0} \int_{\hat{C}_\varepsilon^-(x)} \psi(\xi) \hat{\mu}_\Gamma^\varepsilon(\xi) = \int_{\hat{C}_0^-(x)} \psi(\xi) \hat{\mu}_\Gamma^0(\xi) \quad (\text{C.6})$$

where $\hat{\mu}_\Gamma^\varepsilon(\xi)$ and $\hat{\mu}_\Gamma^0(\xi)$ are the volume forms on $\hat{C}_\varepsilon^-(x)$ and $\hat{C}_0^-(x)$ induced by μ^ε and μ^0 respectively.

However, as we have already observed $\hat{C}_0^-(x)$ is *not* the limit of $\hat{C}_\varepsilon^-(x)$ since it does not contain the part generated by geodesics which get closer to the axis than εR_0 , and therefore contains a tear which destroys the S^2 topology. We now consider the portion of the null cone generated by such geodesics (which restores the S^2 topology). For a fixed value of γ the geodesics which generate the missing piece are given by those which satisfy the initial condition that β is proportional to ε . Since the geodesics which satisfy $r_{\min} > \varepsilon R_0$ tend to straight lines as $\varepsilon \rightarrow 0$, when written in quasi-Cartesian coordinates, the previously excluded geodesics must satisfy $\tan \beta < \varepsilon R_0/a$ (for sufficiently small ε). We therefore consider the surface $S_\varepsilon(P)$ generated by past directed null geodesics $\kappa_{s,\gamma}^\varepsilon(t)$ which emanate from the point P (which we may assume has coordinates $(0, a, 0, 0)$) making polar angles

$$\gamma \in (0, \pi), \quad \beta = s\varepsilon R_0/a \quad s \in [-1, 1]$$

and then consider the limit of this surface as $\varepsilon \rightarrow 0$.

Let D_ε be the region for which $r < \varepsilon R_0$. Then the geodesics which generate S_ε leave the point P , enter the region D_ε at $\kappa_{s,\gamma}^\varepsilon(t_0^\varepsilon)$ where $t_0^\varepsilon = -a/\sin \gamma + O(\varepsilon)$, emerge from D_ε at $\kappa_{s,\gamma}^\varepsilon(t_1^\varepsilon)$ where $t_1^\varepsilon = -a/\sin \gamma + O(\varepsilon)$, and then move outwards to infinity. By the previous analysis the two portions outside D_ε both tend to straight lines as $\varepsilon \rightarrow 0$, while the two points $\kappa_{s,\gamma}^\varepsilon(t_0^\varepsilon)$ and $\kappa_{s,\gamma}^\varepsilon(t_1^\varepsilon)$ both tend to the point on the axis with Cartesian coordinates $(-a/\sin \gamma, 0, 0, -a \cot \gamma)$. Thus as $\varepsilon \rightarrow 0$ such geodesics tend to a straight line with a kink in it as it passes through the axis. In Cartesian coordinates $\kappa_{s,\gamma}^0(t)$ is therefore given by

$$\left. \begin{aligned} x &= a + t \sin \gamma \\ y &= 0 \\ z &= t \cos \gamma \end{aligned} \right\} \quad -a/\sin \gamma \leq t \leq 0$$

$$\left. \begin{aligned} x &= -(a + t \sin \gamma) \cos \delta_s \\ y &= -(a + t \sin \gamma) \sin \delta_s \\ z &= t \cos \gamma \end{aligned} \right\} \quad t \leq -a/\sin \gamma$$

where δ_s is some scattering angle which depends upon s .

For $s = 1$ we have a geodesic which grazes D_ε , so that its deflection will be given by that of a geodesic in the conical space-time, and hence $\delta_1 = -\alpha$. Similarly $\delta_{-1} = \alpha$. As γ varies between zero and π and s varies between -1 and 1 , then the first part of the geodesic fills in the gap in $\hat{C}_0^-(P)$ caused by geodesics which hit the axis, while the second part generates a portion of the surface

$$x^2 + y^2 = (a - \sqrt{t^2 - z^2})^2, \quad t \leq -|z|$$

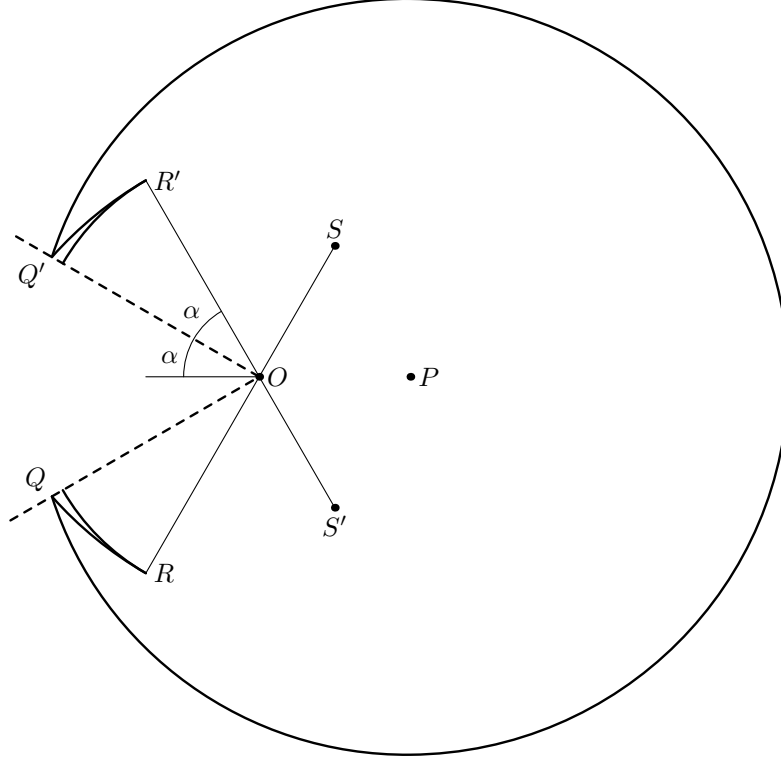


Figure 2.

with edges given by $y = \pm x \tan \alpha$ which exactly match with the tear in $\hat{C}_0^-(P)$ and restore the S^2 topology of the null cone. See Figure 2 for a section of the null cone through $t = \text{constant}$, $z = \text{constant}$.

It is important to note that without imposing further conditions on the regularisation (such as the scalar curvature of g_ε being non-negative) we cannot assume that δ_s is a monotonic function of s or that it lies in the range $-\alpha \leq \delta_s \leq \alpha$. However we do know that the boundary is given by $\delta_1 = -\alpha$ and $\delta_{-1} = \alpha$. Therefore if we consider an integral of a scalar field over the parametrised surface then (apart from a set of measure zero) each point outside the range occurs an even number of times (with one half having the opposite orientation from the other and so cancelling) and each point within the range occurs an odd number of times (with all but one occurrence cancelling). So that if we integrate a scalar field over this region we need only integrate over the region $\tilde{C}_0^-(P)$ given by

$$x^2 + y^2 = (a - \sqrt{t^2 - z^2})^2, \quad t \leq -|z|, \quad -x \tan \alpha \leq y \leq x \tan \alpha$$

which is independent of the regularisation.

It is more natural to include the 2-surface generated by the first part of the geodesic in the definition of $\hat{C}_0^-(P)$, so that we can remove the restriction on geodesics hitting the axis and $\hat{C}_0^-(P)$ is now simply defined to be the past null cone of P in $(M \setminus \Lambda)$. However (C.6) remains valid, and combining this with the above we obtain for all $\psi \in \mathcal{D}(M)$

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{C_\varepsilon^-} (x) \psi(\xi) \mu_\Gamma^\varepsilon(\xi) &= \int_{\hat{C}_0^-} (x) \psi(\xi) \hat{\mu}_\Gamma^0(\xi) + \int_{\tilde{C}_0^-} (x) \psi(\xi) \tilde{\mu}_\Gamma^0(\xi) \\ &= \int_{C_0^-} (x) \psi(\xi) \mu_\Gamma^0(\xi) \end{aligned} \tag{C.7}$$

where $C_0^-(x) = \hat{C}_0^-(x) \cup \tilde{C}_0^-(x)$ and μ_Γ^0 is the volume form induced by g_0 on $\hat{C}_0^-(x)$ and $\tilde{C}_0^-(x)$. Note that points on the axis are excluded from both $\hat{C}_0^-(x)$ and $\tilde{C}_0^-(x)$, but do not contribute to the integral on the left hand side either due to the (uniform) boundedness of μ^ε even when points on the axis are included.

The final point we have to deal with is to show that $\Lambda^-(x)$ and $\mu_{\Lambda^-}^0$ is well defined. $\Lambda^-(x)$ is defined to be the limit of $C_\varepsilon^-(x) \cap \Lambda$ as $\varepsilon \rightarrow 0$. As we have seen $\kappa_{s,\gamma}^\varepsilon(t_\varepsilon^0)$ tends to the point $(-a/\sin \gamma, 0, 0, -a \cot \gamma)$ as $\varepsilon \rightarrow 0$. So that as γ varies between zero and π we see that $C_\varepsilon^-(P) \cap \Lambda$ is given by a curve of the form

$$x = 0, \quad y = 0, \quad t^2 - z^2 = a^2 + n_\varepsilon(t, z), \quad t < 0 \quad (\text{C.8})$$

where $n_\varepsilon \rightarrow 0$ uniformly as $\varepsilon \rightarrow 0$. In the limit we obtain $\Lambda^-(x)$ which is given by

$$x = 0, \quad y = 0, \quad t^2 - z^2 = a^2, \quad t < 0 \quad (\text{C.9})$$

which is independent of the regularisation.

Also for all values of ε the volume form induced on Λ by g_ε is given by the 2-dimensional Minkowskian value μ_2 , (since the singular part is orthogonal to the axis and the smoothing leaves the metric parallel to the axis unchanged). Now by (C.9) $\Lambda^-(x)$ is a smooth curve so the volume form $\mu_{\Lambda^-}^0$ induced on it by μ_2 is well defined and is indeed the limiting value of the volume form induced on (C.8).

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